On the Complete Classification of Competitive Three-Dimensional Gompertz Model for Nullcline Stability[☆]

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Abstract

We investigate the model of three competitive species, each of which, in isolation, admits Gompertz growth. A well-known theorem by M.W.Hirsch guarantees the existence of carrying simplex. Based on this, we compare three dimensional competitive Gompertz models with three dimensional competitive Lotka-Volterra models, and we find that each Gompertz model has a corresponding Lotka-Volterra model with identical nullclines. We then present the complete classification for nullcline stability and arrive at the total of 33 stable nullcline classes, and show that in 27 of these classes all the compact limit sets are fixed points. Despite the common results, we go on to show that the behavior on the carrying simplex of Gompertz systems is subtly different from that on Lotka-Volterra systems. The number of limit cycles is finite in 5 of the remaining 6 classes, and that only the classes 26 and 27 admit Hopf bifurcations and the other 4 do not. The class 27, which has a heteroclinic polycycle, contains a system to have May-Leonard phenomenon: the existence of nonperiodic oscillation and still admits one to have at least two

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limit cycles. The numerical stimulation reveals that there are some systems in class 28 with two limit cycles.

Keywords: competitive system, carrying simplex, Gompertz model, classification, Hopf bifurcation, nonperiodic oscillation

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1. Introduction

There is an extensive literature in population ecology on deterministic models of the Kolmogorov form

$$\frac{dx_i}{dt} = x_i f_i(x_1, x_2, ..., x_n), \quad 1 \leq i \leq n, \ x_i \geq 0, \tag{1}$$

where x_i represents the population density of the ith species and $f_i(x)$ represents the *per capita* growth rate of the ith species. The system (1) is called *competitive* if $f(x)$ is continuously differentiable and $\frac{\partial f_i}{\partial x_j} \leq 0$ on the closed positive cone $\mathbf{R}^n_+$ for $i \neq j$, and *totally competitive* if $\frac{\partial f_i}{\partial x_j} < 0$ on $\mathbf{R}^n_+$ for all i, j. Much of the literature on Kolmogorov system (1) has focused on competitive systems. Smale [1] showed that any vector field on the standard $(n-1)$-simplex in $\mathbf{R}^n$ can be embedded in a smooth totally competitive system on $\mathbf{R}^n_+$, for which the simplex is an attractor. Hirsch [2] proved that every positive limit set lies in an invariant open $(n-1)$-cell, and that the flow in any limit set of system (1) is conjugate to the flow in some invariant set of a Lipschitz vector field in $\mathbf{R}^{n-1}$. Hirsch [3] showed that if system (1) is totally competitive and dissipative on $\mathbf{R}^n_+$ with the origin as a repeller, then every nontrivial trajectory is asymptotic to one in Σ, where Σ is homeomorphic to the standard $(n-1)$-simplex Δ^{n-1} by radial projection. According to Zeeman [4], the Σ is called *carrying simplex.* This theory is very powerful for three-dimensional competitive system : Smith [5] proved Poincaré-Bendixson Theorem holds for three-dimensional competitive systems, and Hirsch [6] and Smith [7] provided the classification for limit sets of three-dimensional competitive systems in some sense.

In ecology, the most frequently used model is the Lotka-Volterra system, that is, each per capita growth function f_i is affine and chosen as the logistic growth. In this circumstance, system (1) reads as

$$\frac{dx_i}{dt} = x_i(r_i - \sum_{j=1}^{n} a_{ij}x_j), \quad 1 \leq i \leq n, \; x_i \geq 0. \tag{2}$$

The system (2) is a totally competitive system if all parameters r_i, a_{ij} are positive. The set CLV(3) of all these three dimensional competitive Lotka-Volterra systems corresponds to parameter space $\text{int}\mathbf{R}_+^{12}$ one to one. Based on the theory of the carrying simplex, Zeeman [4] used a geometric analysis of nullclines of a Lotka-Volterra system to define a combinatorial equivalence relation on the space, named *nullcline equivalence*, by simple algebraic inequalities on the parameters. A vector field $F \in \text{CLV}(3)$ is said to be *nullcline stable* if its equivalence class is an open set in CLV(3). In this remarkable paper, Zeeman arrived at exactly 33 stable nullcline classes, and showed that in 25 of these classes there are no periodic orbits, whose dynamics are fully described; she also proved that Hopf bifurcations occur in each of six stable nullcline classes among the remaining eight classes but not in the other two. Van den Driessche and Zeeman [8] ruled out periodic orbits of the last two classes. This is due to Zeeman's fundamental classification theory, and many researchers have investigated multiplicity of limit cycles, see Hofbauer and So [9], Xiao and Li [10], Lu and Luo [11, 12], Gyllenberg and Yan [13], Gyllenberg, Yan and Wang[14]. Classification results for other three-dimensional competitive systems in this spirit were provided by Li and Smith [15] and van den Driessche and Zeeman [16].

To be notable, the logistic growth is not suitable for some populations, while a lot of works, such as Burton [17], Laird [18], Simpson-Herren and Lloyd [19], Steel [20] and Sullivan and Salmon [21], showed the suitability of the Gompertz growth law (that is, the per capita growth rate is the logarithm $\ln\frac{K}{x}$ developed by Gompertz (1825) which was derived from the actuarial model) to tumor growth. They got the results mainly by curve fitting with actual data. The Gompertz law shows a better data fit than other growth laws such as the logistic model when tumor data involves a wide range of sizes [20]. It seems that no strong biological or physical argument can interpret the reason that the Gompertz model fits actual tumor data primely. Gompertz equation ever aided in the design of successful clinical trials [22, 23] though the frequent use is empirical and based upon data fitting, and it is often accepted in the study of cancer and the therapy. For survey article we refer [24]. Universally we have almost applied Gompertz to represent the growth of microorganisms [25, 26] and some creature [27, 28]. A lot of research on mobile communications has been carried out with particular emphasis on their diffusion at national (Botelho Pinto, 2004) as well as at international level (Fildes Kumar, 2002; Gruber, 2005) during recent years. Note that the Gompertz function reaches the maximum rate of growth at an earlier phase than the logistic, it is the best chosen to show the dynamics of the diffusion process whose growth is so rapid at an early phase while slow relatively when approaching the saturation level. It becomes visible that Gompertz model is appropriate enough for precise fitting and predicting the diffusion of mobile telephony in this case, see [29, 30, 31, 32, 33]. When the competition is among several regions or countries, it is reasonable to model multidimensional Gompertz equations. Yu, Wang and Lu [34] proposed the three-dimensional Gompertz model

$$\begin{aligned}
\frac{dx_1}{dt} &= x_1 \ln \frac{b_1}{x_1 + a_{12}x_2 + a_{13}x_3}, \\
\frac{dx_2}{dt} &= x_2 \ln \frac{b_2}{a_{21}x_1 + x_2 + a_{23}x_3}, \\
\frac{dx_3}{dt} &= x_3 \ln \frac{b_3}{a_{31}x_1 + a_{32}x_2 + x_3},
\end{aligned} \tag{3}$$

and then they analyzed the existence of local stability of all equilibria, ruled out the existence of nontrivial periodic solutions and obtained global stability for some cases.

This paper will present the complete classification of nullcline stability for competitive three-dimensional Gompertz models by using the idea of Zeeman [4]. We compare three dimensional competitive Gompertz models with three dimensional competitive Lotka-Volterra models, and we find that each Gompertz model has a corresponding Lotka-Volterra model with identical nullclines. There are exactly 33 stable equivalence classes of three dimensional Gompertz models, in 27 of which all their trajectories converge to fixed points. All fixed points are hyperbolic except the interior fixed point in classes 26 and 27. Despite the common results, we go on to show that the behavior on the carrying simplex of Gompertz systems is subtly different from that on Lotka-Volterra systems. We shall prove that only two classes (classes 26 and 27) can occur Hopf bifurcation. It is shown that the number of limit cycles of system (3) is finite if it has not any heteroclinic polycycle in $\mathbf{R}^3_+$. In the class admitting a heteroclinic polycycle (class 27), we provide the criteria for the interior fixed point to be globally asymptotically stable and for the system to possess May-Leonard phenomenon: the existence of nonperiodic oscillation, and exhibit the results to bifurcate one or two limit cycles. The numerical stimulation reveals that there are some systems in class 28 with two limit cycles. Whether the classes $29, 30$ and 31 have limit cycles or not remains open. The maximum number of limit cycles that occur in each of classes 26 to 31 remains open.

2. Classification by nullcline equivalence

Consider an n-dimensional competitive Gompertz system

$$\frac{dx_i}{dt} = x_i \ln \frac{b_i}{\sum_{j=1}^{n} a_{ij} x_j}, \quad 1 \leq i \leq n,\ x_i \geq 0, \tag{4}$$

where $a_{ij}, b_i > 0, i, j = 1, 2, ..., n$.

We define $R(0) = \{x \in \mathbf{R}_+^n : \alpha(x) = 0\}$ and $\Sigma = \partial R(0) \setminus R(0)$, where $\alpha(x)$ denotes the alpha limit set of x and $\partial R(0)$ denotes the boundary of $R(0)$ taken in $\mathbf{R}_+^n$. The unit simplex in $\mathbf{R}_+^n$ is defined to be $\Delta^{n-1} = \{x \in \mathbf{R}_+^n : \sum_{i=1}^{n} x_i = 1\}$.

Proposition 2.1. *Given system* (4)*, every trajectory in* $\mathbf{R}_+^n \setminus \{0\}$ *is asymptotic to one in* Σ*, and* Σ *is a Lipschitz submanifold, homeomorphic to* Δ^{n-1} *in* $\mathbf{R}_+^n$ *by radial projection.*

Proof. The proof is just like that of Lotka-Volterra in [4], the only difference is that $\partial f_i / \partial x_j = -a_{ij} (\sum_{j=1}^{n} a_{ij} x_j)^{-1}$ at any $x \neq 0$ for all i, j, where $f_i(x) = \ln(b_i / \sum_{j=1}^{n} a_{ij} x_j)$. □

Denote by

$$\mathrm{CG}(n) = \{F : \mathbf{R}_+^n \to \mathbf{R}^n, F_i(x) = x_i \ln \frac{b_i}{\sum_{j=1}^{n} a_{ij} x_j},\ a_{ij}, b_i > 0,\ 1 \leq i, j \leq n\}$$

the space of n-dimensional competitive Gompertz systems. Let $A = (a_{ij})$ with $a_{ii} = 1, i = 1, 2, ..., n$.

2.1. Classification for the three-dimensional model

Let $F \in \mathrm{CG}(n)$. The nullclines of the system $\dot{x} = F(x)$ are given by

$$\dot{x}_i = 0 \Leftrightarrow x_i \ln \frac{b_i}{(Ax)_i} = 0 \Leftrightarrow (Ax)_i = b_i \text{ or } x_i = 0.$$

Now the classification program in [4] carry over to the Gompertz model here in a straightforward way, that is this part works just like Lotka-Volterra systems, as in this reference, so we don't need to re-do it.

For the two-dimensional case, the nullcline configuration of a vector field $F \in \mathrm{CG}(2)$ is given by the values of $\mathrm{sgn}((AR_i)_j - b_j)$ for $i \neq j$ modulo permutation of the indices, and F and G are said to be nullcline equivalent if they have the same nullcline configurations, where $F, G \in \mathrm{CG}(2)$.

Proposition 2.2. *Let $F \in \mathrm{CG}(2)$. F is nullcline stable* $\Leftrightarrow$ $\mathrm{sgn}((AR_i)_j - b_j) \neq 0,\ for\ i \neq j$.

Corollary 2.3. *There are* 3 *stable nullcline classes and they have open dense union in* $\mathrm{CG}(2)$.

Figure 1: The behavior on the carrying simplex Σ replaced by Δ^1 of the two-dimensional system. A fixed point is represented by a closed dot $\bullet$ if it attracts on Σ, by an open dot $\circ$ if it repels on Σ.

We utilize the phase portrait to describe these stable nullcline classes and in Fig. 1 we list the dynamics on Σ replaced by Δ^1 of a representative from each class.

For the three-dimensional case, the nullcline configuration of a vector field $F \in \mathrm{CG}(3)$ is given by the values of $\mathrm{sgn}((AR_i)_j - b_j)$ and $\mathrm{sgn}((AQ_i)_i - b_i)$ for distinct i, j, modulo permutation of the indices, and F and G are said to be nullcline equivalent if they have the same nullcline configurations, where $F, G \in \mathrm{CG}(3)$.

Proposition 2.4. *Let $F \in \mathrm{CG}(3)$. F is nullcline stable* $\Leftrightarrow$ $\mathrm{sgn}((AR_i)_j - b_j),\ \mathrm{sgn}((AQ_i)_i - b_i) \neq 0,\ for\ i \neq j$.

Proposition 2.5. *If $F \in \mathrm{CG}(3)$ is nullcline stable, then all fixed points on $\partial\Sigma$ are hyperbolic, which implies any interior fixed point in $\mathbf{R}^3_+$, if it exists, is simple.*

The proposition ensures that only P or a closed orbit or a heteroclinic cycle can be the limit set in $\mathrm{int}\Sigma$.

Proposition 2.6. *Let $F \in \mathrm{CG}(3)$ be nullcline stable. Suppose that the axial fixed points R_i, the planar fixed points Q_k and the interior fixed point P exist, with index $\mathrm{I}(R_i), \mathrm{I}(Q_k)$ and $\mathrm{I}(P)$ respectively on Σ, we set $\mathrm{I}(Q_k) = 0$ if the planar fixed point Q_k does not exist, and $\mathrm{I}(P) = 0$ if the interior fixed point P does not exist. Then*

$$\mathrm{I}(R_1) + \mathrm{I}(R_2) + \mathrm{I}(R_3) + 2(\mathrm{I}(Q_1) + \mathrm{I}(Q_2) + \mathrm{I}(Q_3)) + 4\mathrm{I}(P) = 1.$$

Theorem 2.7. *There are totally* 33 *stable nullcline classes in* CG(3).

To be notable, Theorem 2.7 is proved by counting all the combinatorial possibilities for the non-zero values of $\mathrm{sgn}((AR_i)_j - b_j)$ and $\mathrm{sgn}((AQ_k)_k - b_k)$ modulo permutation of the indices, which needs care since these values are dependent. The six values of $\mathrm{sgn}((AR_i)_j - b_j)$ form 2^6 possibilities and then reduce to 16 possibilities modulo permutation of the indices. The values of $\mathrm{sgn}((AR_i)_j - b_j)$ correspond to the ordering of the N_i intercepts of the axial, and then ensures which planar fixed points Q_k lie in $\mathbf{R}^3_+$. Using the index formula in Proposition 2.6, we count 57 possibilities for the non-zero values of $\mathrm{sgn}((AR_i)_j - b_j)$ and $\mathrm{sgn}((AQ_k)_k - b_k)$ concerning the planar fixed points and the interior fixed point, and then reduce to 45 possibilities modulo permutation of the indices. Then we arrive at the total of 33 stable nullcline classes by ruling out all nonexistence cases, which is the same as that of the three-dimensional competitive Lotka-Volterra systems.

Despite the common results, there are details such as the dynamics on the simplex that differ, so we again list them in Fig. 2 and Fig. 3 to be more explicit, which display all the dynamical properties clearly. In Fig. 3 we can see that the dynamics on the simplex of class 28 to class 31 is indeed different from that of Lotka-Volterra systems and the details will be given in next section.

We mark the interior fixed point P (if exists) in $\mathrm{int}\Sigma$ and by the intersection of its hyperbolic manifolds in classes 19 to 25, where P is a saddle on Σ, while by the symbol $\odot$ in classes 26 and 27 since the nullcline configuration does not contain enough information to determine the dynamical type of P, and it also indicates that there may be any number of periodic orbits surrounding P. Propositions 3.12 and 3.13 will show that there are no nontrivial periodic orbits in classes 32 and 33 and P is hyperbolic in classes 28 to 33, which is a repeller on Σ in classes 28, 30 and 32 (see Propositions 3.8, 3.10 and 3.12) and an attractor in classes $29, 31$ and 33 (see Propositions 3.9, 3.11 and 3.13), respectively. So we in advance mark the clear global

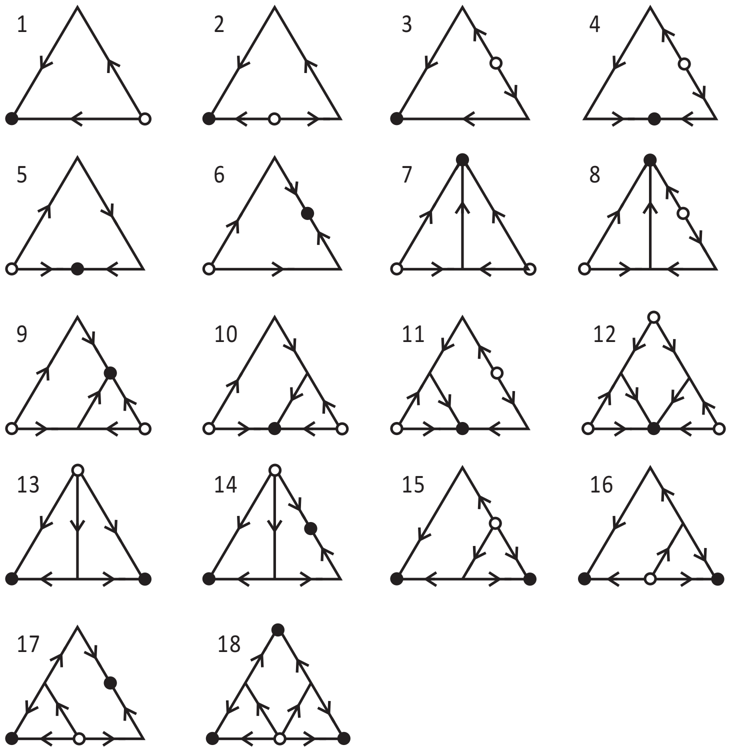

Figure 2: The phase portraits on Σ of three-dimensional stable nullcline classes without interior fixed point. The fixed-point notation is as in Fig. 1.

dynamics in classes 32 and 33 and local dynamics in classes 28 to 31 in Fig. 3, where the big circle ◯ denotes a region of unknown dynamics, which also indicates that there may exist some periodic orbits here.

3. Hopf bifurcations

3.1. Algebraic observations

Now, we are concerned with the behavior on the carrying simplex of each class. Without loss of generality, we assume that $b_i = 1$ for $i = 1, 2, 3$. Otherwise, we scale system (3) by the linear transformation $y_i = x_i/b_i$.

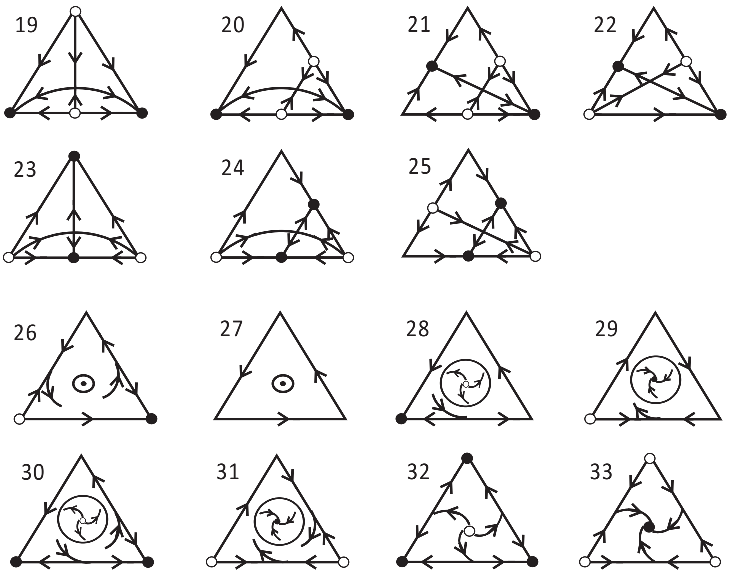

Figure 3: The phase portraits on Σ of three-dimensional stable nullcline classes with interior fixed point.

Then the system (3) has the form:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \ln \frac{1}{x_1 + a_{12}x_2 + a_{13}x_3}, \\ \frac{dx_2}{dt} &= x_2 \ln \frac{1}{a_{21}x_1 + x_2 + a_{23}x_3}, \\ \frac{dx_3}{dt} &= x_3 \ln \frac{1}{a_{31}x_1 + a_{32}x_2 + x_3}. \end{aligned} \tag{5}$$

Now the system is completely determined by the matrix $A = (a_{ij})$ which can be denoted by $A(a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32})$. We shall abuse notation by using system (5): $\dot{x} = F(x)$ and A interchangeably.

The Poincaré-Bendixson theory to the flow on Σ immediately implies the following proposition.

Proposition 3.1. *There are no periodic orbits in stable nullcline classes* 1 *to* 25.

We shall ambiguously use 1 to denote the vector $(1, 1, 1)^\tau$ and the real number 1.

Now, if the system A has an interior fixed point $P = (p_1, p_2, p_3)$, then we have $AP^\tau = 1$, since it satisfies the equations $Ax^\tau = 1$, where $x = (x_1, x_2, x_3)$.

Let $D = \det(A)$ and D_i denote the determinant of the corresponding matrix by changing the ith column of A by 1, and set

$$\alpha_i = 1 - a_{i(i+1)}, \quad \beta_j = a_{j(j+2)} - 1, i, j \in \{1, 2, 3\},$$

in which the induces are understood by mode 3.

By Proposition 2.5, $D \neq 0$, then we have $p_i = D_i/D$, and it is easy to verify that -1 is an eigenvalue of variational matrix of F at P with associated eigenvector $(p_1, p_2, p_3)^\tau$. Hence, we assume that -1, λ_1, λ_2 are the three eigenvalues of DF_P. A routine computation yields $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D$, and one can see [34] for more.

Proposition 3.2. *For every system A from any of the nullcline classes 26 to 33, $D > 0$.*

The proof is similar as that of Lotka-Volterra systems in [4], so we omit it here.

Now we present the sufficient and necessary condition for Hopf bifurcation to occur in classes 26 to 33. Since the proof here also shows a method to construct systems with limit cycles, we give it clearly.

Proposition 3.3. *Hopf bifurcations occur in the stable nullcline class i ($26 \leq i \leq 33$) if and only if there exists a system A with the property: $\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 = 0$.*

Proof. First, we note that $a_{ij} \neq 1$ for $i \neq j$ from a series of Propositions in next two subsections.

Assume that Hopf bifurcations occur in the stable nullcline class i ($26 \leq i \leq 33$). Given a system A from class i, recall that $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D$, so the necessity is obvious.

Conversely, suppose that there exists a system A from class i satisfying $\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 = 0$, i.e., $(a_{13}-1)(a_{21}-1)(a_{32}-1)-(1-a_{12})(1-a_{23})(1-a_{31}) = 0$. We will show that Hopf bifurcations occur in the stable nullcline class i.

It is clear that at least two of α_1, α_2, α_3 are with the same sign, say α_1, α_2. Let P be the interior fixed point of A, and -1, λ_2, λ_3 be the eigenvalues of DF_P. Note that $\lambda_2 + \lambda_3 = 0$ and $\lambda_2\lambda_3 > 0$, thus $(\lambda_2 + \lambda_3)^2 - 4\lambda_2\lambda_3 < 0$. That is, λ_2, λ_3 form a purely imaginary complex-conjugate pair. Since λ_2, λ_3 are continuous with respect to a_{ij}, there is a $\mu_1 > 0$ such that for $|\mu| < \mu_1$, the eigenvalues of the variational matrix $DF^{\mu}_{P^\mu}$ of $A_\mu(a_{12}, a_{13}, a_{21}, a_{23}, a_{31}+\mu, a_{32})$ at P^μ satisfy $(\lambda^\mu_2 + \lambda^\mu_3)^2 - 4\lambda^\mu_2\lambda^\mu_3 < 0$, that is, $\lambda^\mu_2, \lambda^\mu_3$ are a simple pair of complex-conjugate eigenvalues. Recall that A is nullcline stable, then there is a neighborhood $\mathcal{U}$ of A such that for $\forall B \in \mathcal{U}$, B is stable and nullcline equivalent to A. Thus there exists a μ_0 satisfying $0 < \mu_0 < \mu_1$ such that for $|\mu| < \mu_0$, A_μ is stable and nullcline equivalent to A, that is, A_μ is in class i and $DF^{\mu}_{P^\mu}$ has a simple pair of complex conjugate eigenvalues $\lambda(\mu), \overline{\lambda(\mu)}$ and the other -1.

Let $g(\mu) = \beta^\mu_1\beta^\mu_2\beta^\mu_3 - \alpha^\mu_1\alpha^\mu_2\alpha^\mu_3$, $h(\mu) = D^\mu$. Clearly, for any $|\mu| < \mu_0$, $h(\mu) > 0$, and $h(0) = D$, $g(0) = 0$. Actually,

$$\begin{aligned} g(\mu) =& \ \beta^\mu_1\beta^\mu_2\beta^\mu_3 - \alpha^\mu_1\alpha^\mu_2\alpha^\mu_3 \\ =& \ (a_{13}-1)(a_{21}-1)(a_{32}-1) - (1-a_{12})(1-a_{23})(1-a_{31}-\mu) \\ =& \ (1-a_{12})(1-a_{23})\mu. \end{aligned}$$

Set $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, thus $\alpha(\mu) = \frac{g(\mu)}{2h(\mu)}$. So we have $\alpha(0) = g(0)/2h(0) = 0$ and $\alpha'(\mu) = (g'(\mu)h(\mu) - g(\mu)h'(\mu))/2h^2(\mu)$. Then

$$\alpha'(0) = \frac{g'(0)h(0) - g(0)h'(0)}{2h^2(0)} = \frac{(1-a_{12})(1-a_{23})}{2D} > 0.$$

Now we can apply Hopf bifurcation theorem to obtain the conclusion. □

Proposition 3.4. *Let system A be nullcline stable. Suppose the parameters a_{ij} satisfy the following inequalities*

$$a_{ij}a_{ji} \geq 1(\leq 1), \quad a_{ij}a_{jk} + a_{ji}a_{ik} - a_{ik} - a_{jk} \geq 0(\leq 0), \tag{6}$$

where i, j, k are distinct, then the system has no nontrivial periodic orbit.

Proof. We consider two possibilities:

(I) All the inequalities in (6) are equalities, i.e.,

$$a_{ij}a_{ji} = 1, \quad a_{ij}a_{jk} + a_{ji}a_{ik} - a_{ik} - a_{jk} = 0.$$

(II) There exists a strict inequality in (6).

If (I) holds, by assumption of nullcline stability for A, $a_{ij} \neq 1$ for $i \neq j$, then we get $a_{ij} = a_{ik}a_{kj}$. (Here i, j, k are distinct.) Now one can easily check that the corresponding system for this case is in the class 1 (see Fig. 2), hence the system has no nontrivial periodic orbit.

If (II) holds, note that Yu, Wang and Lu [34] have given a criterion for nonexistence of periodic orbits by the method of Busenberg and van den Driessche (see [35, 8]). In the proof of Theorem 3.1 of [34], they actually verified that if the strict inequality

$$\begin{aligned}&[\tfrac{1}{x_3}(\tfrac{a_{21}-1}{a_{21}x_1+x_2+a_{23}x_3}+\tfrac{a_{12}-1}{x_1+a_{12}x_2+a_{13}x_3})\\&+\tfrac{1}{x_2}(\tfrac{a_{31}-1}{a_{31}x_1+a_{32}x_2+x_3}+\tfrac{a_{13}-1}{x_1+a_{12}x_2+a_{13}x_3})\\&+\tfrac{1}{x_1}(\tfrac{a_{32}-1}{a_{31}x_1+a_{32}x_2+x_3}+\tfrac{a_{23}-1}{a_{21}x_1+x_2+a_{23}x_3})]>0(<0)\end{aligned} \tag{7}$$

holds, then the system has no nontrivial periodic orbit (see (3.11) in [34]). By calculation, it is not difficult to see the above strict inequality holds in this case.

Since the possibilities (I) and (II) are exhaustive, the proposition is proved. □

3.2. Families with Hopf bifurcations

In this subsection we shall show that Hopf bifurcations occur in the stable nullcline classes 26 and 27, which is the same as that on Lotka-Volterra models. See [4].

Proposition 3.5. *A system is in the nullcline class* 26 *if and only if* a_{ij} *satisfy*

(i) $a_{12} > 1, a_{13} < 1, a_{21} < 1, a_{23} > 1, a_{31} < 1, a_{32} > 1,$

(ii) $a_{12}(1-a_{23}) + a_{13}(1-a_{32}) + a_{23}a_{32} - 1 < 0,$ *and*

(iii) $a_{21}(1-a_{13}) + a_{23}(1-a_{31}) + a_{13}a_{31} - 1 < 0,$

where a_{ij} *are given by modulo permutation of the indices. The stable nullcline class* 26 *admits a Hopf bifurcation and, consequently, periodic orbits.*

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 26.

Let A be a given system from nullcline class 26. Set

$$\begin{aligned}g_1(A) &= a_{12}(1-a_{23}) + a_{13}(1-a_{32}) + a_{23}a_{32} - 1,\\ g_2(A) &= a_{21}(1-a_{13}) + a_{23}(1-a_{31}) + a_{13}a_{31} - 1,\\ g(A) &= (a_{13}-1)(a_{21}-1)(a_{32}-1) - (1-a_{12})(1-a_{23})(1-a_{31}).\end{aligned}$$

Let the parameter a_{12} vary and the others be fixed. Then g_1 is decreasing with respect to a_{12} and g_2 is constant. So as $a_{12} \to \infty$, the new parameters

a_{ij} satisfy (i), (ii), (iii) all the same, i.e., the new system A given by a_{ij} is still in nullcline class 26, and $g(A) \to -\infty < 0$. First, we choose some a_{12} sufficiently large, such that system $A = (a_{ij})$ is in nullcline class 26, and $g(A) < 0$.

Note that, with respect to a_{31}, g_2 is decreasing and g_1 is constant. Then based on the system A chosen by last step, we let $a_{31} \to 1$, the new parameters a_{ij} always satisfy (i), (ii) and (iii), i.e., the new system A given by a_{ij} is still in nullcline class 26, and we have $g(A) \to (a_{13}-1)(a_{21}-1)(a_{32}-1) > 0$. Since $g(A)$ is continuous with respect to a_{31}, there exists some $a_{31} < 1$, such that $A = (a_{ij})$ is in nullcline class 26 satisfying $g(A) = 0$, i.e., $\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 = 0$. Now the existence of Hopf bifurcation follows from Proposition 3.3. □

Proposition 3.6. *A system is in the nullcline class* 27 *if and only if* a_{ij} *satisfy*

$$a_{12} > 1, a_{13} < 1, a_{21} < 1, a_{23} > 1, a_{31} > 1, a_{32} < 1,$$

where a_{ij} *are given by modulo permutation of the indices. The stable nullcline class* 27 *admits a Hopf bifurcation and, consequently, periodic orbits.*

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 27.

Let $g(A) = (a_{13} - 1)(a_{21} - 1)(a_{32} - 1) - (1 - a_{12})(1 - a_{23})(1 - a_{31})$. Note that given a system $A = (a_{ij})$ from the stable nullcline class 27, for all $a_{31} > 1$, the system is always in nullcline class 27. Thus let $a_{31} \to 1$, we have $g(A) \to (a_{13} - 1)(a_{21} - 1)(a_{32} - 1) < 0$, and let $a_{31} \to \infty$, we have $g(A) \to \infty > 0$. Since $g(A)$ is continuous with respect to a_{31}, there exists some $a_{31} > 1$, such that $A = (a_{ij})$ is in nullcline class 27 with the property: $g(A) = 0$, i.e., $\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 = 0$. The existence of Hopf bifurcation follows from Proposition 3.3. □

We now note that any system in the class 27 has a heteroclinic cycle $R_1 \to R_2 \to R_3 \to R_1$ of May-Leonard type (respectively with the arrows reversed). Let λ_{ij} be the external eigenvalue at the equilibrium R_i in the direction j which is given by $-\ln a_{ji}$, and set

$$p_\lambda := \lambda_{12}\lambda_{23}\lambda_{31} + \lambda_{13}\lambda_{32}\lambda_{21} = -(\ln a_{12} \ln a_{23} \ln a_{31} + \ln a_{21} \ln a_{32} \ln a_{13}). \quad (8)$$

According to the results in [36], we have the following proposition.

Proposition 3.7. *The heteroclinic cycle is stable when* $p_\lambda < 0$ *and the heteroclinic cycle is unstable when* $p_\lambda > 0$.

Remark 3.1. Using (6), we can present a system in the class 27 that has *May and Leonard phenomenon*: the system exhibits a general class of solutions with non-periodic oscillations of bounded amplitude but ever-increasing cycle time; asymptotically, the system cycles from being composed almost wholly of population 1, to almost wholly 2, to almost wholly 3, back to almost wholly 1 etc. In mathematical language, almost all ω limit sets are the boundary of the carrying simplex Σ. See the Example 4.4 in the next section.

3.3. Families without Hopf bifurcations

In this subsection, we shall show that within each of the stable nullcline classes 28 to 31 there are no Hopf bifurcations, and any system in classes 32 and 33 has no periodic orbits, in which all compact limit sets are fixed points.

Proposition 3.8. *A system is in the nullcline class* 28 *if and only if* a_{ij} *satisfy*

(i) $a_{12} > 1, a_{13} < 1, a_{21} > 1, a_{23} > 1, a_{31} > 1, a_{32} < 1,$
(ii) $a_{31}(1 - a_{12}) + a_{32}(1 - a_{21}) + a_{12}a_{21} - 1 < 0,$

where a_{ij} *are given by modulo permutation of the indices. The interior fixed point* P *of each system from class* 28 *is hyperbolic and a repeller on* Σ, *hence there are no Hopf bifurcations within nullcline class* 28.

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 28.

By the conditions (i) and (ii), we have $(a_{13} - 1)(a_{21} - 1)(a_{32} - 1) > 0$, $(1 - a_{12})(1 - a_{23})(1 - a_{31}) < 0$, thus $(a_{13} - 1)(a_{21} - 1)(a_{32} - 1) - (1 - a_{12})(1 - a_{23})(1 - a_{31}) > 0$, i.e., $\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 > 0$. Recall that $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D$, then both of λ_2 and λ_3 have positive real part, where λ_2, λ_3 are the other two eigenvalues of DF_P except -1. So the interior fixed point P of each system from class 28 is hyperbolic and a repeller when restricted to Σ and hence within nullcline class 28 there is no Hopf bifurcation. □

Remark 3.2. This is subtly different from that on Lotka-Volterra systems, for which there exists Hopf bifurcation in class 28. Moreover, even in the subset those with $a_{ii} = 1$ of Lotka-Volterra models corresponding to Gompertz models there also exists Hopf bifurcation. See [4]. While Example 4.2 in the following shows that it is possible to have limit cycles in the class 28.

Proposition 3.9. *A system is in the nullcline class* 29 *if and only if* a_{ij} *satisfy*

(i) $a_{12} < 1, a_{13} > 1, a_{21} < 1, a_{23} < 1, a_{31} < 1, a_{32} > 1,$
(ii) $a_{31}(1 - a_{12}) + a_{32}(1 - a_{21}) + a_{12}a_{21} - 1 < 0,$

where a_{ij} *are given by modulo permutation of the indices. The interior fixed point* P *of each system from class* 29 *is hyperbolic and always locally asymptotically stable, hence within nullcline class* 29 *there are no Hopf bifurcations.*

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 29.

By the conditions (i) and (ii), we have $(a_{13} - 1)(a_{21} - 1)(a_{32} - 1) < 0$, $(1 - a_{12})(1 - a_{23})(1 - a_{31}) > 0$, thus $(a_{13} - 1)(a_{21} - 1)(a_{32} - 1) - (1 - a_{12})(1 - a_{23})(1 - a_{31}) < 0$, i.e., $\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 < 0$. Recall that $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D$, then both of λ_2 and λ_3 have negative real part. So the interior fixed point P of each system from class 29 is hyperbolic and always locally asymptotically stable. Thus, there are no Hopf bifurcations among nullcline class 29 . □

Remark 3.3. This is subtly different from that on Lotka-Volterra systems, for which there exists Hopf bifurcation in class 29. Moreover, even in the subset those with $a_{ii} = 1$ of Lotka-Volterra models corresponding to Gompertz models there also exists Hopf bifurcation. See [4].

Proposition 3.10. *A system is in the nullcline class* 30 *if and only if* a_{ij} *satisfy*

(i) $a_{12} > 1, a_{13} < 1, a_{21} > 1, a_{23} > 1, a_{31} > 1, a_{32} > 1,$
(ii) $a_{12}(1 - a_{23}) + a_{13}(1 - a_{32}) + a_{23}a_{32} - 1 < 0,$
(iii) $a_{31}(1 - a_{12}) + a_{32}(1 - a_{21}) + a_{12}a_{21} - 1 < 0,$

where a_{ij} *are given by modulo permutation of the indices. The interior fixed point* P *of each system from class* 30 *is hyperbolic and a repeller on* Σ, *hence within nullcline class* 30 *there are no Hopf bifurcations.*

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 30.

First, we claim that $a_{12} > a_{32}$. Otherwise, then by condition (i) we have

$$\begin{aligned}&a_{12}(1 - a_{23}) + a_{13}(1 - a_{32}) + a_{23}a_{32} - 1\\ &\geq a_{32}(1 - a_{23}) + a_{13}(1 - a_{32}) + a_{23}a_{32} - 1\\ &= (a_{32} - 1)(1 - a_{13}) > 0,\end{aligned}$$

contradicting to (ii). Then the claim holds, which implies that

$$a_{23} > \frac{a_{12} + a_{13} - a_{32}a_{13} - 1}{a_{12} - a_{32}}.$$

Thus

$$a_{23} - 1 > \frac{a_{12} + a_{13} - a_{32}a_{13} - 1}{a_{12} - a_{32}} - 1 = \frac{(a_{32} - 1)(1 - a_{13})}{a_{12} - a_{32}}.$$

By condition (iii), we have

$$a_{31} - 1 > \frac{a_{32} - 1 + a_{12}a_{21} - a_{32}a_{21}}{a_{12} - 1} - 1 = \frac{(a_{21} - 1)(a_{12} - a_{32})}{a_{12} - 1}.$$

Now

$$\begin{aligned} & (a_{13} - 1)(a_{21} - 1)(a_{32} - 1) + (a_{12} - 1)(a_{23} - 1)(a_{31} - 1) \\ > \; & (a_{13} - 1)(a_{21} - 1)(a_{32} - 1) + (1 - a_{13})(a_{21} - 1)(a_{32} - 1) = 0. \end{aligned}$$

Recall that $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D$, then both of λ_2 and λ_3 have positive real part. So the interior fixed point P of each system from class 30 is hyperbolic and a repeller on Σ and hence within nullcline class 30 there are no Hopf bifurcations. $\square$

Remark 3.4. This is subtly different from that on Lotka-Volterra systems, for which there exists Hopf bifurcation in class 30. Moreover, even in the subset those with $a_{ii} = 1$ of Lotka-Volterra models corresponding to Gompertz models there also exists Hopf bifurcation. See [4].

Proposition 3.11. *A system is in the nullcline class* 31 *if and only if* a_{ij} *satisfy*

(i) $a_{12} < 1, a_{13} > 1, a_{21} < 1, a_{23} < 1, a_{31} < 1, a_{32} < 1,$

(ii) $a_{12}(1 - a_{23}) + a_{13}(1 - a_{32}) + a_{23}a_{32} - 1 < 0,$

(iii) $a_{31}(1 - a_{12}) + a_{32}(1 - a_{21}) + a_{12}a_{21} - 1 < 0,$

where a_{ij} *are given by modulo permutation of the indices. The interior fixed point* P *of each system from class* 31 *is hyperbolic and locally asymptotically stable, hence within nullcline class* 31 *there are no Hopf bifurcations.*

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 31.

First, by conditions (i) and (ii) we have

$$a_{13} < \frac{a_{12} - 1 + a_{32}a_{23} - a_{12}a_{23}}{a_{32} - 1},$$

then

$$a_{13} - 1 < \frac{a_{12} - 1 + a_{32}a_{23} - a_{12}a_{23}}{a_{32} - 1} - 1 = \frac{(a_{23} - 1)(a_{12} - a_{32})}{1 - a_{32}}.$$

And by conditions (i) and (iii),

$$a_{31} - 1 < \frac{a_{32} - 1 + a_{12}a_{21} - a_{32}a_{21}}{a_{12} - 1} - 1 = \frac{(a_{21} - 1)(a_{12} - a_{32})}{a_{12} - 1}.$$

Now

$$\begin{aligned} & (a_{13} - 1)(a_{21} - 1)(a_{32} - 1) + (a_{12} - 1)(a_{23} - 1)(a_{31} - 1) \\ < \; & -(a_{23} - 1)(a_{21} - 1)(a_{12} - a_{32}) + (a_{23} - 1)(a_{21} - 1)(a_{12} - a_{32}) = 0. \end{aligned}$$

Recall that $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D$, then both of λ_2 and λ_3 have negative real part. So the interior fixed point P of each system from class 31 is hyperbolic and locally asymptotically stable, and hence within nullcline class 31 there are no Hopf bifurcations. $\square$

Remark 3.5. This is subtly different from that on Lotka-Volterra systems, for which there exists Hopf bifurcation in class 31. Moreover, even in the subset those with $a_{ii} = 1$ of Lotka-Volterra models corresponding to Gompertz models there also exists Hopf bifurcation. See [4].

Proposition 3.12. *A system is in the nullcline class* 32 *if and only if* a_{ij} *satisfy*

(i) $a_{12} > 1, a_{13} > 1, a_{21} > 1, a_{23} > 1, a_{31} > 1, a_{32} > 1,$
(ii) $a_{12}(1 - a_{23}) + a_{13}(1 - a_{32}) + a_{23}a_{32} - 1 < 0,$
(iii) $a_{21}(1 - a_{13}) + a_{23}(1 - a_{31}) + a_{13}a_{31} - 1 < 0,$
(iv) $a_{31}(1 - a_{12}) + a_{32}(1 - a_{21}) + a_{12}a_{21} - 1 < 0,$

where a_{ij} *are given by modulo permutation of the indices. The stable nullcline class* 32 *has no nontrivial periodic orbits, and the interior fixed point is hyperbolic and a repeller on* Σ*, every trajectory converges to one of the fixed points* R_i*,* $i \in \{1, 2, 3\}$*.*

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 32.

By the condition (i) and Proposition 3.4, we obtain that any system in the stable nullcline class 32 has no nontrivial periodic orbits. It is easy to see that

$$\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 = (a_{13}-1)(a_{21}-1)(a_{32}-1) - (1-a_{12})(1-a_{23})(1-a_{31}) > 0,$$

i.e., $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D > 0$. Thus P is hyperbolic and a repeller on Σ, and by Poincaré-Bendixson theory we have every trajectory converges to one of the fixed points R_i, $i \in \{1, 2, 3\}$. □

Proposition 3.13. *A system is in the nullcline class* 33 *if and only if* a_{ij} *satisfy*

(i) $a_{12} < 1, a_{13} < 1, a_{21} < 1, a_{23} < 1, a_{31} < 1, a_{32} < 1,$
(ii) $a_{12}(1 - a_{23}) + a_{13}(1 - a_{32}) + a_{23}a_{32} - 1 < 0,$
(iii) $a_{21}(1 - a_{13}) + a_{23}(1 - a_{31}) + a_{13}a_{31} - 1 < 0,$
(iv) $a_{31}(1 - a_{12}) + a_{32}(1 - a_{21}) + a_{12}a_{21} - 1 < 0,$

where a_{ij} *are given by modulo permutation of the indices. The stable nullcline class* 33 *has no nontrivial periodic orbits, hence the interior fixed point* P *is hyperbolic and globally asymptotically stable in* $\mathrm{int}\mathbf{R}^3_+$.

Proof. The algebraic inequalities amongst the parameters can be translated into easily by the nullcline configuration of the stable nullcline class 33.

From the condition (i) and Proposition 3.4 any system in the stable nullcline class 33 has no nontrivial periodic orbits. It is easy to see that

$$\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3 = (a_{13}-1)(a_{21}-1)(a_{32}-1) - (1-a_{12})(1-a_{23})(1-a_{31}) < 0,$$

i.e., $\lambda_2 + \lambda_3 = (\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\alpha_3)/D < 0$. Thus P is hyperbolic and an attractor, and by Poincaré-Bendixson theory, it is globally asymptotically stable in $\mathrm{int}\mathbf{R}^3_+$. □

Remark 3.6. The behavior on the carrying simplex of class 32 and class 33 is the same as that of Lotka-Volterra models. See [8].

Theorem 3.14. *Suppose a three-dimensional competitive Gompertz system has a nontrivial periodic orbit. Then one of the following holds*
(i) *the number of periodic orbits is finite*;

(ii) *there are countably infinite periodic orbits, which accumulate at a polycycle*;
(iii) intΣ *is composed of nontrivial periodic orbits except the interior fixed point* P.
The cases (ii) *and* (iii) *only occur in the class* 27.

Proof. The proof for the Lotka-Volterra systems in [10] carry over to the Gompertz models here in a straightforward way. □

We see that if system (5) has not any heteroclinic polycycles, then the number of limit cycles of system (5) is finite, hence the number of limit cycles of system (5) is finite except nullcline class 27. In the class 27, the number of limit cycles is still finite if $p_\lambda \neq 0$.

4. Some examples

In this section, we investigate bifurcations and *May and Leonard phenomenon* of some three-dimensional competitive Gompertz systems. In Example 4.2, we obtain a family of systems with at least one limit cycle, and we have conducted numerical experiments to survey the properties of the limit cycles. We find that as the parameter varying, the system changes from class 27 to class 28 and there exist at least two limit cycles. In Example 4.3, we construct a family of systems of class 27 with at least two limit cycles. In Example 4.4, we construct two systems with a heteroclinic polycycle to be the global attractor, which shows that there exist systems in the class 27 to possess *May and Leonard phenomenon*.

Example 4.1. Let

$$A = \begin{pmatrix} 1 & 4 & \frac{3}{4} \\ \frac{1}{4} & 1 & \frac{5}{4} \\ \frac{15}{16} & \frac{5}{4} & 1 \end{pmatrix}.$$

It is easy to verify that system A is in the nullcline class 26 with the interior fixed point $P = (\frac{4}{17}, \frac{1}{17}, \frac{12}{17})$, and $-1, \frac{\sqrt{30}}{34}i, -\frac{\sqrt{30}}{34}i$ are the three eigenvalues of matrix DF_P. Now set

$$A(\mu) = \begin{pmatrix} 1 & 4 & \frac{3}{4} \\ \frac{1}{4} & 1 & \frac{5}{4} \\ \frac{15}{16} + \mu & \frac{5}{4} & 1 \end{pmatrix},$$

we know that there exists some $\mu_0 > 0$ such that for $|\mu| < \mu_0$, system $A(\mu)$ is always in the nullcline class 26, and $DF^{\mu}_{P\mu}$ has a simple pair of complex-conjugate eigenvalues $\lambda(\mu), \overline{\lambda(\mu)}$. Let $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, then $\alpha(\mu) = 3\mu/(\frac{85}{4} + 34\mu)$. Clearly, $\alpha(0) = 0$, and $\alpha'(0) = \frac{12}{85} > 0$, hence it is a family of systems given by $A(\mu)$ within class 26 admitting Hopf bifurcations.

Example 4.2. Now consider system

$$A = \begin{pmatrix} 1 & 2 & \frac{1}{2} \\ \frac{11}{19} & 1 & \frac{21}{20} \\ 5 & \frac{1}{20} & 1 \end{pmatrix}.$$

It is easy to verify that system A is in the nullcline class 27 with the interior fixed point $P = (\frac{19}{259}, \frac{80}{259}, \frac{160}{259})$, and $-1, \frac{4\sqrt{458}}{259}i, -\frac{4\sqrt{458}}{259}i$ are the three eigenvalues of matrix DF_P. Now set

$$A(\mu) = \begin{pmatrix} 1 & 2 & \frac{1}{2} \\ \frac{11}{19} + \mu & 1 & \frac{21}{20} \\ 5 & \frac{1}{20} & 1 \end{pmatrix},$$

we know that there exists some $\mu_0 > 0$ such that for $|\mu| < \mu_0$, system $A(\mu)$ is always in the nullcline class 27, and $DF^{\mu}_{P\mu}$ has a simple pair of complex-conjugate eigenvalues $\lambda(\mu), \overline{\lambda(\mu)}$. Let $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, then $\alpha(\mu) = 19\mu/(\frac{59311}{95} - 158\mu)$. Clearly, $\alpha(0) = 0$, and $\alpha'(0) = \frac{1805}{59311} > 0$, hence it is a family of systems given by $A(\mu)$ within class 27 admitting Hopf bifurcations. To determine the stability properties of these periodic orbits, it is enough to study the stability of $P = (\frac{19}{259}, \frac{80}{259}, \frac{160}{259})$ of system A, which can be reduced to the study of the corresponding equations on a center manifold of P.

By calculating we obtain that the singular point P of system A is a stable focus with the first negative focal value -1.547628679. Since the focal value is a rather lengthy expression, the exact value was computed as a *rational* by computer. Now by Hopf bifurcation theorem, we know that system $A(\mu)$ admits a stable limit cycle when $0 < \mu \ll 1$. And recall the phase portrait of class 27, the system always has a heteroclinic polycycle with three saddles. It is easy to check that $p_\lambda = -\ln(2)\ln(\frac{21}{20})\ln(5) - \ln(2)\ln(20)\ln(\frac{11}{19} + \mu) > 0$ when $0 < \mu \ll 1$, which determines that the heteroclinic polycycle of system

$A(\mu)$ is unstable. Hence system $A(\mu)$ has at least one limit cycle when $0 < \mu \ll 1$.

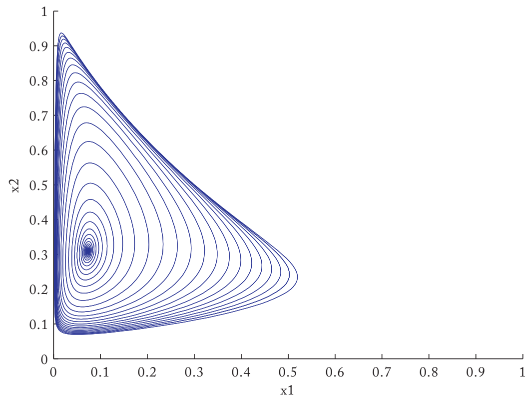

Figure 4: A family of stable limit cycles occur as a_{21} increasing.

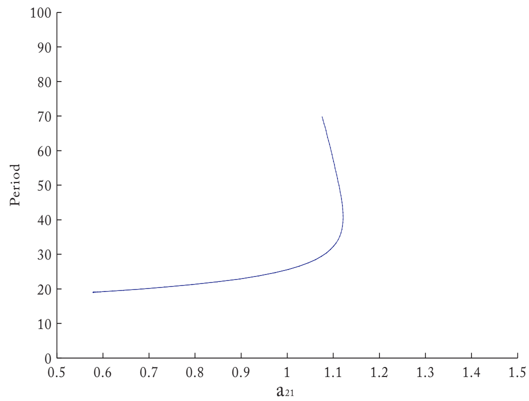

Figure 5: The period of the limit cycles increases as a_{21} increasing, and even if $a_{21} > 1$ the periodic orbit also occurs.

Now we use the graphing capability of Matlab [37, 38] to illustrate the properties of the limit cycles as the parameter μ varying in this example. Take $\mu = 0$ as the initial value, with which the system is in the nullcline class 27. As μ increasing, i.e., a_{21} increasing, we see that the system A_μ has a stable limit cycle with the period increasing. See Fig. 4 and Fig. 5. And in Fig. 5 we find that the periodic orbit also occurs even if $a_{21} > 1$, and in some interval the system which is now in the nullcline class 28 has at least two limit cycles.

Example 4.3. Consider the system

$$A = \begin{pmatrix} 1 & 200 & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 \\ \frac{1593}{1592} & \frac{1}{2} & 1 \end{pmatrix}.$$

It is easy to verify that system A is in the nullcline class 27 with the interior fixed point $P = (\frac{796}{1195}, \frac{1}{1195}, \frac{398}{1195})$, and $-1, \frac{\sqrt{18164592242}}{571210}i, -\frac{\sqrt{18164592242}}{571210}i$ are the three eigenvalues of matrix DF_P. Now let

$$A(\mu) = \begin{pmatrix} 1 & 200 & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 \\ \frac{1593}{1592} + \mu & \frac{1}{2} & 1 \end{pmatrix},$$

we know that there exists some $\mu_0 > 0$ such that for $|\mu| < \mu_0$, system $A(\mu)$ is always in the nullcline class 27, and $DF^{\mu}_{P^{\mu}}$ has a simple pair of complex-conjugate eigenvalues $\lambda(\mu), \overline{\lambda(\mu)}$. Let $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, then it is easy to see that $\alpha(0) = 0$ and $\alpha'(0) > 0$. Hence it is a family of systems given by $A(\mu)$ within class 27 admitting Hopf bifurcations. To determine the stability properties of these periodic orbits, it is enough to study the stability of $P = (\frac{796}{1195}, \frac{1}{1195}, \frac{398}{1195})$ of system A, which can be reduced to the study of the corresponding equations on a center manifold of P.

By calculating we obtain that the singular point P of system A is an unstable focus with the first positive focal value 0.04578207028. Since the focal value is a rather lengthy expression, the exact value was computed as a *rational* by computer. Now by Hopf bifurcation theorem, we know that system $A(\mu)$ admits an unstable limit cycle when $-1 \ll \mu < 0$. And recall the phase portrait of class 27, the system always has a heteroclinic polycycle with three saddles. It is easy to check that $p_\lambda = -\ln(200)\ln(2)\ln(\frac{1593}{1592} + \mu) + \ln^3(2) > 0$ when $-1 \ll \mu < 0$, which determines that the heteroclinic polycycle of system $A(\mu)$ is unstable. So there exists a stable limit cycle between the unstable limit cycle and the heteroclinic polycycle. Hence system $A(\mu)$ has at least two limit cycles when $-1 \ll \mu < 0$.

Example 4.4. Consider the system

$$A = \begin{pmatrix} 1 & 4 & \frac{1}{2} \\ \frac{1}{2} & 1 & 4 \\ 4 & \frac{1}{2} & 1 \end{pmatrix}.$$

It is easy to check that system A is in the nullcline class 27 with the interior fixed point $P = (\frac{2}{11}, \frac{2}{11}, \frac{2}{11})$, and a_{ij} satisfy condition (6), hence by Proposition 3.4 we get system A has no nontrivial periodic orbit. Since $p_\lambda = -7\ln^3(2) < 0$ and $(a_{13}-1)(a_{21}-1)(a_{32}-1)-(1-a_{12})(1-a_{23})(1-a_{31}) = \frac{215}{8} > 0$, P repels and the heteroclinic polycycle is a global attractor.

Now we consider the new system

$$A = \begin{pmatrix} 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 \\ 2 & \frac{1}{2} & 1 \end{pmatrix},$$

which is still in the nullcline class 27 with the interior fixed point $P = (\frac{2}{7}, \frac{2}{7}, \frac{2}{7})$, and it is easy to check that $p_\lambda = 0$. Clearly, a_{ij} satisfy condition (6), hence by Proposition 3.4 we get this system has no nontrivial periodic orbit. Since $(a_{13}-1)(a_{21}-1)(a_{32}-1)-(1-a_{12})(1-a_{23})(1-a_{31}) = \frac{7}{8} > 0$, P repels which implies that the heteroclinic polycycle is also a global attractor. See Fig. 6.

5. Conclusions

This paper has investigated the model of three competitive species, each of which, in isolation, admits Gompertz growth. What we are concerned is its saturation level, i.e., its long-run behavior in mathematical language. Then we compare three dimensional competitive Gompertz models with three dimensional competitive Lotka-Volterra models.

We have presented the complete classification for nullcline stability and arrived at the total of 33 stable nullcline classes which is the same as that of three dimensional competitive Lotka-Volterra systems CLV(3) (see [4]) although CG(3) has six independent parameters, while CLV(3) has eight independent parameters. We have shown that all fixed points are hyperbolic

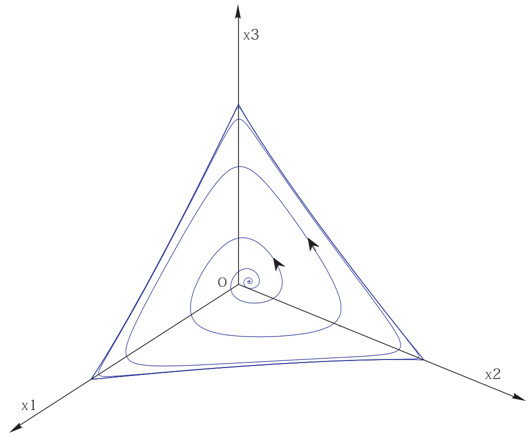

Figure 6: The solution of system A approaches the heteroclinic polycycle.

and all the compact limit sets are fixed points in classes $1-25, 32$ and 33. This makes us to conjecture any system in these 27 classes are structurally stable. Despite the common results, we go on to show that the behavior on the carrying simplex of Gompertz systems is subtly different from that on Lotka-Volterra systems, and even the subset of Lotka-Volterra models (those with $a_{ii} = 1$) corresponding to Gompertz models do not have the same limitations of behavior. As far as periodic solutions are concerned, there are distinct differences between three dimensional Lotka-Volterra systems and Gompertz systems. For Lotka-Volterra systems, nontrivial periodic solution-s only occur in classes 26 to 31 (see [4] and [8]) and Hopf bifurcations do occur in the classes 26 to 31, hence there possibly exist nontrivial periodic solutions in these six classes. However, for Gompertz systems, Hopf bifurcations only occur in classes 26 and 27 and there are no Hopf bifurcations in classes $28-31$ because the interior fixed point is hyperbolic in these four classes. From the phase portraits in classes $28-31$ of Fig. 3, if there exist limit cycles which are hyperbolic, then there are at least two limit cycles. The numerical stimulation reveals that there are some systems in class 28 with at least two limit cycles whose amplitudes are surely not small. We have carried out numerous numerical experiments within classes $29-31$ but havn't found any nontrivial periodic solution. Whether the classes $29, 30$ and 31 have limit cycles or not remains open. In the class 27, there exists a heteroclinic polycycle. Logically, there are four types of asymptotic behavior:

the interior fixed point is globally asymptotically stable; there is at least one limit cycle; the heteroclinic polycycle is globally asymptotically stable; and intΣ is completely filled with periodic orbits. For Lotka-Volterra systems, all four cases happen in the class 27 (see [4] and [39]). Although we have proved this class can bifurcate one and two limit cycles (see Examples 4.2 and 4.3) and provided a criterion for the interior fixed point and the heteroclinic polycycle to be globally asymptotically stable (see Proposition 3.4 and Example 4.4), we cannot give an example such that the interior fixed point is a center on Σ which holds when the heteroclinic polycycle is neutral in symmetric and asymmetric May-Leonard systems (see [39]). However, this result does not hold in class 27 of Gompertz systems (see Example 4.4). It is an open problem whether three dimensional competitive Gompertz systems have a center on Σ. It is proved that the number of limit cycles is finite in classes $26, 28-31$. The maximum number of limit cycles that occur in each of classes 26 to 31 remains open.

Permanence only occur in classes $29, 31, 33$ and 27 with the heteroclinic polycycle unstable.

The parameter conditions on each class can be easily translated from the configuration.

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